

Similarities in the Mathematics of Demography and the Theory of Interest

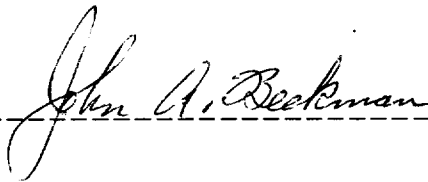
An Honors Thesis (HONRS 499)

by

Edward L. Pyle

Thesis Advisor

John A. Beekman



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Ball State University

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### Purpose of Thesis:

The study of demography is an area of actuarial science that many students do not have an opportunity to study as an undergraduate. The study of a population and its characteristics, particularly its growth, is vital to the actuary. This discussion focuses on demography, particularly the aspect of the growth of a population, and ways in which it relates to the field of actuarial science. One way to examine demography is to compare models of population growth to the growth of a fund over time due to interest. There are a diversity of examples that demonstrate the relationship between population and theory of interest. One important aspect of demography is projections - estimating future populations. The mathematics of projections is similar to the theory of interest and is important to the actuary. In addition, population growth through the course of history is examined, including the views of Thomas Malthus, one of the first to identify negative aspects of population growth.

As long as mankind has been on Earth, its population has been concerned with the growth of its numbers. From primitive times, when each new birth within a group meant an increased chance of survival for the group, to modern times when countries are fighting to lower birth rates, human population has grown astonishingly, especially in the last century. Today, due to overwhelming growth, people are faced with population issues that in the past would have seemed unbelievable.

As Nathan Keyfitz notes, the frame of reference that modern man uses to view population has shifted from tribal to global (13, vii). In the past, any growth in the numbers of an individual group necessarily meant a stronger group, thus enhancing the individual's chances for survival. Each group required births in great numbers to compensate for high death rates that prevailed throughout most of history. In Roman times, more children meant more soldiers with which to conquer new lands (13, vii). Overpopulation could never be a danger to them because the bounds of the earth seemed unlimited.

Since Roman times, however, phenomenal world population growth has forced modern civilization to view birth rates quite differently. A key factor in the incredible growth rates seen in modern times has been an increased expectation of life unchecked by a decrease in birth rates. This is due to greater technology and advances in medicine and nutrition. In recent years, governments have altered their view of high birth rates and in many cases, their policies reflect this recent concern. Due to the problems of overpopulation, world leaders view high birth rates as a detriment to society rather than as an advantage. Today, stories in the news of massive starvation and malnutrition in overpopulated developing countries are not unusual. People hear about damage to the environment due to

rapid population growth. In poor nations, overpopulation causes ever deepening poverty. In these nations, the proportionally large numbers of people under age 15 and over age 65 (those who generally are supported by others) require money for support which could otherwise be used for investing, thus improving economic conditions (12, 30-32). This has caused countries to institute laws and programs in an effort to control such problems. In China, for instance, to overcome this problem, leaders have implemented a policy of 'one couple, one child' (14, 22). Incentives such as better housing and better medical care encourage the Chinese to follow this policy. In addition, those who do not comply are often fined or lose their jobs. Governments and international organizations have focused on population growth as a global problem and have expressed concerns in a variety of ways.

Demography is the study of characteristics of human population, such as size, growth, density and distribution. The growth of a population, therefore, is only a part of demography. However, due to the impact that this aspect of demography has had in recent years on human life, it has been given much of the attention of demographers. Demographers frequently represent the growth of a population by mathematical models. By taking trends evident in a population in the past and extrapolating these trends into the future, demographers can obtain extremely useful predictions of future characteristics of a population. This, in demography, is known as forecasting.

Forecasting is not only useful to demographers hoping to gain an understanding of how to solve population problems. Forecasts are used in many ways and on a variety of scales. Forecasts can be used on a global scale to estimate future needs of resources. They can also be used, on a smaller scale, by individual communities to predict the need in the future of schools or hospitals. Forecasts,

therefore can be extremely helpful in many aspects of modern civilization.

A profession in which forecasts are important is that of the actuary. It is inherent in the job of the actuary to estimate future needs of businesses and institutions. Knowledge of future population characteristics can be an aid in predicting such future needs. For example, with the average age of the population in the United States growing, due to the large numbers born in the Baby Boom, actuaries have dealt increasingly with insurance and pension plans for the elderly. An enormous area of growing insurance needs is long term care. Actuaries, witnessing the Baby Boom, have modified their estimates of future needs of the insurance industry.

An aspect of population growth that is of great interest to actuaries, is the mathematical models that demographers use to forecast future populations. Actuaries use the mathematical concept of interest to estimate future financial needs due to the growth of money. In the study of the growth of a population, mathematical concepts, in many cases, are identical to those used in the accumulation of interest of a fund. The purpose of the following text is to show examples of how questions concerning the growth of a population can easily be converted to questions concerning the accrual of interest.

To begin, Stephen G. Kellison discusses two significant methods of accumulation of interest, simple interest and compound interest (8, 5-8). In demographic theory, a population can grow in a mathematically identical way. It can theoretically grow in an arithmetic pattern or in a geometric pattern as Nathan Keyfitz describes (9, 2-3).

In interest theory, a fund that accumulates by a constant amount of interest is increasing by "simple interest" (8, 5). The amount of a fund, at year  $t$ , accruing

under simple interest is equal to  $A(1+it)$  where  $A$  is the initial amount of the fund and  $i$  is the amount of interest compounded annually. A numerical example follows:

Find the accumulated value of \$20,000 invested for three years if the rate of *simple interest* is 3% per annum (8, 6).

Solution:

$$20,000[1+(.03)(3)] = \$21,800.$$

When a fund accumulates at simple interest, the interest earned is not added to the fund to accrue further interest. The theory of compound interest assumes that the interest earned is automatically reinvested (8, 6). An initial fund equal to  $A$  under compound interest accumulation after  $t$  years will equal  $A(1+i)^t$  where again  $i$  is the rate of interest compounded annually. The symbol,  $^$ , is used to mean "raised to the power." The previous example is reworked under the assumption of compound interest as opposed to simple interest:

Find the accumulated value of \$20,000 invested for 3 years if the rate of *compound interest* is 3% per annum (8, 8).

Solution:

$$\$20,000(1.03)^3 = \$21,855.$$

The total under compound interest is greater because interest earned in each year increases.

Keyfitz discusses how a population can grow either geometrically or arithmetically (9, 2). A population growing arithmetically is the mathematical equivalent of a fund accumulating by simple interest while a population growing geometrically is equivalent to a fund accumulating by compound interest. Instead of  $i$  representing a rate of interest, problems involving the growth of a population

use  $r$ , a rate of population growth.

The previous examples can be reworded to demonstrate problems involving a population growing rather than a fund accumulating interest. Using the same numbers the problem reads:

Find the total population if an initial population of 20,000 people grows *arithmetically* at an annual rate of 3% for 3 years.

Solution:

The solution is equivalent to that of example 1.1

$$20,000[1 + (.03)(3)] = 21,800.$$

Under the assumption of geometric growth the same problem reads:

Find the total population if an initial population of 20,000 people grows *geometrically* at an annual rate of 3% for 3 years:

Solution:

$$20,000(1.03)^3 = 21,855.$$

Compound interest is used much more frequently than is simple interest. Similarly, the model of geometric growth of a population in demography is preferred over the model of arithmetic growth (9, 3). Further discussions will focus on accumulation by compound interest in theory of interest and geometric growth of a population in demography.

In mathematics of finance, there are many situations in which interest is compounded at periods other than annually. When interest is paid more frequently than once a year, the rate of interest is known as the "nominal rate" (9, 14). When interest is payable  $m$  times per year the nominal rate of interest will be denoted  $i^*m$ , or "i upper m." To find the nominal rate that is equivalent to the effective rate payable annually, the formula  $1 + i = [1 + (i^*m)/m]^m$  is used (8, 15). The

mathematics of the accumulation of interest at periods other than annually can be applied to a population growth.

The limit, as  $m$  approaches infinity, of  $i*m$  is "delta", a nominal rate of interest convertible continuously, also called the "force of interest". Applying this to the formula above, the following results:  $1 + i = [1 + (i*m)/m]^m = \exp[\text{delta}]$ . The application of this will be shown by example, first in terms of interest and then in terms of population.

Find the accumulated value of \$10,000 invested for 10 years if the force of interest is 5% (8, 23 example slightly modified).

Solution:

$$\$10,000\exp[(.05)(10)] = \$10,000\exp(.5) = \$16,487$$

When a population is assumed to be increasing continuously, it is said to be growing exponentially. The problem in terms of a population reads:

Find the total population if an initial population of 10,000 grows exponentially for 10 years.

Solution:

$$10,000\exp[(.05)(10)] = 16,487.$$

The period over which a population growth is compounded has a significant effect on the total amount of the population over time. For instance, a population growing at a rate of 5 percent compounded annually, will differ from a population growing at a rate of 5 percent compounded monthly. This can be shown by the following example:

A population growing at annual rate  $r$  compounded continuously reaches a population at time  $t$  equal to  $P(t) = P(0)\exp(rt)$ . At what rate  $r'$ , compounded annually, would it have to grow so as to equal  $P(t)$  at time  $t$ ?



(Express in terms of  $r$ ) (11, 5).

Solution:

$$P(0)\exp(rt) = P(0)(1+r')^t$$

$$\exp(rt) = (1+r')^t$$

$$[\exp(rt)]^{(1/t)} = 1+r'$$

$$r' = \exp(r) - 1$$

Thus, a population growing at a rate of .05 compounded continuously after time  $t$  would have the same total people as a population growing at  $r' = \exp(.05) - 1 = .05127$ . Further, if  $r = .06$ ,  $r' = .06184$ , and if  $r = .07$ ,  $r' = .07251$ .

Over long periods of time, the interval over which growth is compounded produces significant differences. For instance, the following situation:

A population of 1,000,000 increases at 2 percent per year for 100 years.

What difference does it make if the 2 percent is compounded annually or continuously. (11, 5)

Solution:

$$\text{Compounded annually: } 1,000,000(1.02)^{100} = 7,244,646$$

$$\text{Compounded continuously: } 1,000,000\exp[100(.02)] = 7,389,056$$

The difference is 144,410

The rate compounded annually which would equal the total generated through continuous compounding would be:

$$r' = \exp(.02) - 1 = .0202$$

Mathematically equivalent results occur if looking at the above example as a problem of interest. An initial fund of \$1,000,000 generating interest at 2 percent compounded annually would differ after 100 years from the same initial fund generating interest at 2 percent compounded continuously. The total after 100 years

compounded annually is \$7,244,646 and the total after 100 years compounded continuously is \$7,389,056.

One way of examining a population growth is by focusing on its "doubling time", the amount of time it would take a population to double in number if increasing at a certain rate. If increasing at a geometric rate, the doubling time of a population is  $n$  which satisfies the formula:  $(1+r)^n = 2$  (9, 4). If the population is growing exponentially or "compounded continuously" the formula is  $\exp[nr] = 2$ . The following example shows how this concept applies to interest as well as population.

A population growing exponentially stood at 1,000,000 in 1930 and at 3,000,000 in 1970.

(a) What is its rate of increase?

(b) What is its doubling time? (11, 6).

Solution:

$$(a) 1,000,000\exp[40r] = 3,000,000$$

$$\exp[40r] = 3$$

$$40r = \ln(3)$$

$$r = .0275$$

$$(b) \exp[(.0275)n] = 2$$

$$(.0275)n = \ln(2)$$

$$n = 25.2054 \text{ years}$$

The problem as a fund accumulating interest:

An investment compounded continuously stood at \$1,000,000 in 1930 and at \$3,000,000 in 1970.

(a) What is the force of interest?

(b) What is its doubling time?

Solution:

$$(a) \$1,000,000 \exp[40(\delta)] = \$3,000,000$$

$$\delta = .0275$$

$$(b) \exp[(.0275)t] = 2$$

$$t = 25.2054 \text{ years}$$

Similarly, tripling time and other such growth intervals can be found as seen in the following problem.

Write down the simple formula for tripling time at rate of increase  $r$  percent per year compounded continuously (11,8).

Solution:

$$\exp[rt] = 3$$

$$rt = \ln(3)$$

$$t = \ln(3)/r \text{ (or approximately } 1.1/r)$$

As before, this problem also can be converted to find the tripling time of a fund accumulating interest using *delta*, the force of interest, instead of  $r$ , a rate of increase of a population. The result of such a problem would be identical.

In demographic theory, finding the total population after a certain number of years becomes more complicated when the rate of increase changes each year. The following example will demonstrate the first type of variation in which the effective rate of interest changes at a specific period of time.

Find the accumulated value of \$100,000 after 5 years if the effective rate of interest changes annually and the corresponding rates for each year at 2%, 3%, 3%, 2.5% and 2%.

Solution:

First find the accumulated value of \$1 at time  $t$ :

$$a(t) = [1+i(1)][1+i(2)]\dots[1+i(t)]$$

$$a(5) = (1.02)(1.03)(1.03)(1.025)(1.02) = 1.13135$$

The accumulated value of \$100,000 is:

$$\$100,000a(5) = \$113,135$$

If this example was a problem in demography, a population of 100,000 would increase to 113,135 people over 5 years. In this case it is rate of increase of population rather than interest rate which changes annually.

Another type of variation occurs when the rate is continuously changing over time. The following situation demonstrates such a case:

A population starting with 1,000,000 people was growing exponentially at 3.5% per year when it first came under observation. Its rate of growth immediately started to fall by the amount of 0.1% per year. What was the population at the end of 15 years? (11, 9).

Solution:

Use  $P(T) = P_0 \exp\left[\int_0^T r(t)dt\right]$  where  $P(T)$  is the population at time  $T$  and  $P_0$  is the initial population (8, 18).

The rate at time  $t$  is defined as  $r(t) = .035 - .001t$

The population at time  $t$  is  $P(T) = 1,000,000 \exp\left[\int_0^T r(u)du\right]$

The population after 15 years is  $P(15) = 1,000,000 \exp\left[\int_0^{15} (.035 - .001u)du\right]$   
 $= 1,000,000 \exp\left[(.035u - .0005u^2)\right]_0^{15}$

$$= 1,000,000 \exp[0.4125]$$

$$= 1,510,589.5 \text{ (or approximately 1,510,590 people)}$$

Converted to a question involving varying force of interest, the problem uses the formula 1.23,  $a(t) = \exp\left[\int_0^t \delta(r)dr\right]$  (8, 24).

An investment starting with \$1,000,000 increased at a force of interest of 3.5% at the time of investment. Its force of interest immediately started to fall by the amount of 0.1% per year. What was the amount of the fund after 15 years?

Solution:

Use  $A(T) = xa(t)$  where  $x$  is the initial amount of the investment and  $a(t)$  is as defined above.

$$(\delta \text{ sub } t) = .035 - .001t$$

The amount of the fund at time  $t$  is  $A(t) = \$1,000,000 \exp\left[\int_0^t (\delta \text{ sub } r) dr\right]$

At year 15,  $A(15) = \$1,510,589.5$ .

A similar example follows:

The rate of increase of a population at time  $t$  is  $r(t) = .01 + 0.0001t^2$ . If the population totals 1,000,000 at time zero, what is it at time 30? (11, 9)

Solution:

$$1,000,000 \exp\left[\int_0^{30} (.01 + .0001u^2) du\right]$$

$$= 1,000,000 \exp[1.2]$$

$$= 3,320,116.9 \text{ (or approximately 3,320,117 people).}$$

Like the previous example, this problem can be reworded to involve varying force of interest instead of varying rate of increase:

The force of interest of a fund at time  $t$  is  $(\delta \text{ sub } t) = .01 + 0.0001t^2$ . If the initial fund is \$1,000,000, what is the total accumulated after 30 years?

Solution:

$$1,000,000 \exp\left[\int_0^{30} (.01 + .0001s^2) ds\right]$$

$$= \$3,320,116.9$$

The growth of a population can in many ways be viewed as a parallel to the

growth of a fund due to interest. The types of demographic problems reviewed can easily be converted to problems in the theory of interest and vice versa. Mathematically, the problems are solved identically. Mathematical parallels can be seen not only in growth, but also in such matters as doubling time and varying rates over time.

Projections in demography are used for many purposes and at many levels. A typical use for a projection is in planning. For instance, at a national level, a population projection would be useful in determining the level of demand for governmental services (16, 180). On a smaller scale, a town may be interested in estimating future demographics to determine the need for a new school or a hospital. Estimates of future populations can be made using the mathematics as discussed above. Using rates of increase one can easily derive population forecasts. Using known statistics, demographers estimate the future statistics based on current assumptions and expectations. An example of a forecast follows.

The official U.S. population estimate for mid-1965 was 194,303,000; for mid-1970 was 204,879,000. Extrapolate to 1975 assuming geometric increase (11, 5).

Solution:

First find the rate of increase:

$$194,303,000(1+r)^5 = 204,879,000$$

$$(1+r)^5 = 1.05443$$

$$r = .010656$$

Use this rate to project into the future year, 1975

$$204,879,000(1.010656)^5 = x$$

$$x = 216,030,086$$

Using more recent figures, the population for years 2000 and 2050 are forecasted below:

The population of the United States in 1980 was 226,545,805 and in 1990 it was 248,709,873 (18, 7).

Solution:

Forecast the population of the United States in the year 2000:

$$226,545,805(1+r)^{10} = 248,709,873$$

$$r = .009378$$

$$226,545,805(1.009378)^{10} = x$$

$$x = 272,863,085$$

Forecast to the year 2050

$$226,545,805(1.009378)^{60} = x$$

$$x = 435,147,532$$

If assumptions behind a projection are wrong, then the forecast of the future will be wrong. Because of this, forecasts tend to be more accurate for the near future and become progressively worse as the length of time grows (16, 181). For the type of forecast seen in the above example, the assumption of a constant rate of growth over a long period of time will almost certainly be wrong. Thus this type of projection may be reasonable for a few years, but over time, it will lead to ridiculous figures (16, 182). J. Douglas Faires and Barbara T. Faires show what can happen when forecasting under the assumption of a constant rate of increase.

Estimates of the U.S. population for certain years from 1930 to 1980 are as follows:

1930 - 123,203,000

1940 - 131,669,000

1970 - 203,212,000

1980 - 226,505,000

Predict the U.S. population in the year 2000 on the basis of the population in years:

a) 1930 and 1940

b) 1970 and 1980 (7, 430)

Solution:

$$a) 123,203,000(1+r)^{10} = 131,669,000$$

$$(1+r)^{10} = 1.06872$$

$$r = 0.006668$$

$$131,669,000(1.006668)^{60} = x$$

$$x = 196,182,084$$

$$b) 203,212,000(1+r)^{10} = 226,505,000$$

$$(1+r)^{10} = 1.114624$$

$$r = .010911$$

$$226,505,000(1.010911)^{20} = x$$

$$x = 281,407,870$$

Since the rate of growth between 1930 and 1940 differs from the rate of growth between 1970 and 1980, the two forecasts to the year 2000 are vastly different. The forecast based on more recent data appears more reliable. The result in part a), based on population figures from 1930 and 1940, predicts a population of 196,182,256 in the year 2000, a result that had been surpassed by the year 1970.

As can be seen in the above example, the reason population estimations are more reliable when based upon more recent data is because of the assumption of a constant growth rate. The forecast based upon data from 1930 and 1940 in the above



example assumed that between 1940 and 2000 the rate of growth would remain constant at a rate of .6668%. As was seen in part b), the rate of growth between 1970 and 1980 was 1.0911%. While seemingly small, this variance in growth rates over a period of time produces greatly different results. Like rates of population, interest rates in math of finance cannot be assumed constant. This creates problems for the actuary similar to those of the demographer when making forecasts. Much as the demographer must rely on more recent data, it is essential that the actuary and others in the financial field use current interest rates when making calculations.

The above discussion has shown that there are many mathematical similarities between demography and calculations involving interest. This comparison, however, has examined mathematical form only. Demography is more than just a mathematical science parallel to what actuaries commonly use in math of finance. There are many links for an actuary between demography and financial calculations. John A. Beekman describes a few such ideas from demography which are useful to an actuary (3, 271-276). Some of these concepts involve a relationship between population and financial security, as Keyfitz describes (10, 203). As an example, methods have been suggested in which actuaries can "utilize their techniques to offset the insecurities produced by population waves" (4, 9).

Beekman suggests a relationship between the concept of a stable population in demography and pension mathematics (3, 271-272). A population is 'stable' when mortality and fertility rates remain constant for a long period and there is no migration. This situation leads to a fixed age structure (16, 120). A stable population is one in which an age distribution is unchanging, not the total size of a population. The total population could be growing or shrinking, but the distribution of people among various ages remains constant. A 'stationary' population is a special case of a

stable population that is neither growing nor shrinking, its rate of growth is zero (16, 120). To show how this concept in demography can be useful to actuaries, Beekman uses the following example:

Contrast the financial burden of a social security pension scheme in a country with a stationary population with the financial burden in a country with a stable population with a growth rate of  $r$ .

The burden of old-age pensions can be defined as the ratio of the population over age 65 to the population aged 20 to 65 (11, 59)

Solution:

Let  $R_z(65)$  be the burden of old-age pensions at year  $z$ .

For a stationary population  $R_z(65) = [\int_{65}^w l(y)dy / \int_{20}^{65} l(y)dy]$

For a stable population where births grow exponentially as population

grows.  $R_z(65) = [\int_{65}^w \exp(-ry)l(y)dy / \int_{20}^{65} \exp(-ry)l(y)dy]$

For the stable population, take the logarithm and differentiate with respect to  $r$ :

$$d \ln R / dr = m_1 - m_2$$

$m_1$  is the mean of the population aged 20 to 65

$m_2$  is the mean of the population over age 65

Thus,  $(\text{change in } R) / R = (m_1 - m_2)(\text{change in } r)$

Suppose  $m_1 = 35$  and  $m_2 = 75$ :  $(\text{change in } R) / R = -40(\text{change in } r)$

Thus, for each percentage point of difference in  $r$ , there will be a difference of 40% in  $R$  in the other direction.

For instance, if country A is increasing at 2% per annum and country B is stationary,  $m_1 = 35$  and  $m_2 = 75$ , A's burden is only 20% of B's burden.

An actuary may need to analyze the future financial burden of a pension

system as described in the example above. The actuary will need to have a concept of how population is growing, because the financial need of a population described by a stable population model will be very different from the need of a population described by a stationary population model. Choosing as accurate a model of population growth as possible is important in analyzing future financial needs, an important role of the actuary. As seen in the above problem, two populations growing at different rates will have very different needs in terms of pension plans for retirement.

The demographic problems examined to this point have allowed for perpetual growth of population. However, it is unrealistic to assume that populations can grow without cease. There is only so much space that can be occupied and only so much food that can be produced. One type of model which takes this into account is a 'logistic'. In this model there is an ultimate population,  $a$ , such that the limit, as  $t$  approaches infinity, of  $P(t) = a$  (11, 2). This ultimate population, also called a population ceiling, is the limit to which the population can grow. Thus, as the population approaches  $a$ , there is a slowing down effect in population growth.

There is a simple way to calculate the population ceiling if population totals are known at times  $t(1)$ ,  $t(2)$  and  $2t(2) - t(1)$ , times that are equidistant (9, 22). For an example, the population ceiling is calculated below for the state of Indiana using population totals from years 1820, 1900 and 1980.

The formula for such a calculation is:

$$a = \{ [1/p(1) + 1/p(3) - 2/p(2)] / [1/p(1)p(3) - 1/p(2)^2] \} (11, 3)$$

The populations, are as follows:

1820 - 147,178

1900 - 2,516,462

1980 - 5,490,224 (1, 7 - Table 1)

Solution:

$$a = 147,178^{(-1)} + 5,490,462^{(-1)} - 2 \cdot 2,516,462^{(-1)}$$

$$(147,178 \cdot 5,490,224)^{(-1)} - 2,516,462^{(-2)}$$

$$= 5,725,815$$

[Note: this formula is only applicable if both the numerator and the denominator are positive (11, 3)]

As Keyfitz notes, those who developed the concept of the logistic realized that a population is always growing towards a population ceiling, but due to technological advances and social changes, the ceiling itself can change (9, 22). The population ceiling for the state of Indiana is calculated below using recent data.

1940 - 3,427,796

1960 - 4,662,498

1980 - 5,490,224 (1, 7 - Table 1)

Solution:

$$a = 3,427,796^{(-1)} + 5,490,224^{(-1)} - 2 \cdot 4,662,498^{(-1)}$$

$$(3,427,796 \cdot 5,490,224)^{(-1)} - 4,662,298^{(-2)}$$

$$= 6,294,633$$

The population ceiling using more recent population data is much higher than the ceiling using data from 1820.

In a related problem, one might wish to fit population data into a logistic equation. To derive an equation for  $P(t)$  for the logistic, differential equations are used. When  $P(t) = P(0)\exp(rt)$ ,  $dP(t)/dt = rP(t)$ . This equation allows for continuous fixed growth of a population. For the logistic equation, there is an additional

'slowing-down factor'. Thus, for the logistic,  $dp(t)/dt = rP(t)[1 - P(t)/a]$ , where again  $a$  is the population ceiling (9, 21). When  $P(t) = a$ , the growth rate then is equal to zero. Solving this differential equation for  $P(t)$  leads to the equation  $P(t) = a/[1 + \exp(-r*(t-t_0))]$ , where  $t_0$  is the abscissa of the midpoint of the logistic curve (11, 2). This equation can be stated in several different ways. For example,  $P(t) = [A + B\exp(-ut)]^{(-1)}$  can be useful for solving problems (11, 3). An example follows:

Fit the populations from 1940 and 1980 into a logistic equation

Solution:

The population ceiling, as found in the above example, equals 6,294,633

$$P(0) = 3,427,796$$

$$P(40) = 5,490,224$$

$$P(\text{infinity}) = 6,294,633$$

Use  $P(t) = [A + B\exp(-t * u)]^{(-1)}$  to get a system of three equations and three unknowns.

$$1) P(0) = [A + B\exp(0*u)]^{(-1)}$$

$$2) P(160) = [A + B\exp(-40*u)]^{(-1)}$$

$$3) P(\text{infinity}) = [A + B\exp(-\text{infinity})] = A^{(-1)}$$

$$\text{From equation 3), } A = 1.58865497 * 10^{(-7)}$$

$$\text{Substitute into equation 1) to get } B = 1.3286715 * 10^{(-7)}$$

$$\text{Finally, substitute into equation 2) to get } u = .04354771$$

$$P(t) = [1.58865497 * 10^{(-7)} + 1.3286715 * 10^{(-7)} \exp(-0.4354771 * t)]^{(-1)}$$

Thus, we have an equation that sets the limit of population growth at 6,294,633 and fits the known data.

Using differential equations is useful for demographers in other ways. For example, the following problem:

Set up a differential equation of the form  $dy/dt = uy$  for a Standard Metropolitan Statistical Area (SMSA). Let  $k$  = population in 1960. Use the data for 1970 and 1980 to determine  $u$ . Determine the appropriate units for  $t$ . Solve for  $y$ . (11, 17 - modified)

Solution:

Let the SMSA be Delaware County in Indiana.

Population of Delaware County

1960 - 110,938

1970 - 129,219

1980 - 128,587 (1, 8 - table 2)

Let  $t = 1$  year

Calculate  $u$ :

$$129219 \exp(10u) = 128587$$

$$u = -.0004903$$

$$dy/dt = uy$$

$$1/y = u dt$$

$$\ln y = ut + C$$

$$y = \exp(ut + C)$$

$$y = K \exp(-.0004903 * t)$$

$$y = 110,938 \exp(-.0004903 * t)$$

Using this formula, we can estimate the population for the year 1990 as well as project to future years.

Compute the population of Indiana using  $y = 110,938 \exp(-.0004903*t)$  for years 1990, 2000, 2010 and 2020

Solution:

$$1990: y = 110,938 \exp (-.0004903 * 30) = 109,318$$

$$2000: y = 110,938 \exp (-.0004903 * 40) = 108,783$$

$$2010: y = 110,938 \exp (-.0004903 * 50) = 108,251$$

$$2020: y = 110,938 \exp (-.0004903 * 60) = 107,722$$

The results using  $u$  calculated from 1960 and 1980 are quite different:

$$u = .0073817$$

$$y = 110,938 \exp (.0073817 * t)$$

$$1990: y = 110,938 \exp (.0073817 * 30) = 138,438$$

$$2000: y = 110,938 \exp (.0073817 * 40) = 149,044$$

$$2010: y = 110,938 \exp (.0073817 * 50) = 160,462$$

$$2020: y = 110,938 \exp (.0073817 * 60) = 172,755$$

Again, the necessity of estimating population growth as accurately as possible is demonstrated.

Throughout most of history, a growing population was viewed as an indication of the growing strength of a group or of mankind as a whole. However, a large population is not always a benefit to the society as a whole. One of the first to acknowledge this fact was Thomas Robert Malthus, an English economist who wrote two controversial essays on population growth. The main point of these essays were that there is a "constant tendency in all animated life to increase beyond the nourishment prepared for it" (15,2). At the time of his writings in the early nineteenth century, development of agriculture had led to substantial increases in world populations. Malthus was one of the first to point out that this growth was a detriment. His writings are based upon three principles. First, "population can not increase without the means of subsistence." Second, "population invariably increases when the means of subsistence are available." Finally, "the superior

power of population can't be checked without producing misery or vice" (19, 19).

Malthus envisions a population model with a cyclical nature. When food is abundant, population can grow without harm to the population. However, when population increase is more than can be matched by increase in food production, misery is inevitable. Malthus points out that in his native England, the growth in population due to improvements in agriculture and manufacturing did not improve "standards of living, or more generally, 'happiness and virtue'" (19, 23). Malthus' writings were controversial because "his ideas lay primarily in the defeat of a long-standing tradition which had automatically linked a large and growing population with economic progress and national power" (19, 1).

Malthus indicated mathematical reasons behind the fact that population tends to outgrow subsistence. He wrote that population grows exponentially, while food can only be increased linearly. A population growing without check will double every 25 years (15, 7). Thus, an initial population would increase "as the numbers 1, 2, 4, 8, 16, 32, 64, 128, 256." Subsistence would grow linearly "as 1, 2, 3, 4, 5, 6, 7, 8, 9." Thus "in 2 centuries, the population would be to the means of subsistence as 256 to 9."

Population, therefore can be compared to a fund growing at interest compounded continuously while subsistence grows as a fund growing under simple interest. Over time, these funds, with equivalent initial amounts, will have significantly different totals.

Malthus said that vice and immoral activity was a check to population growth in times when growth overwhelmed the availability of food. Thus, he endorsed 'moral restraint' as a means of countering such a situation (19, 23). For example, "refraining from or postponing marriage on grounds of the likely effect on



the economic and social status of having to support a family" was a desirable method of facing social problems (19, 23).

In retrospect, historical demographers grant that Malthus "provides a valuable guide to the behavior of preindustrial societies" (19, 95). Since the eighteenth century, however, trends in population demographics would have greatly surprised Malthus. The number of people worldwide has escalated phenomenally since Malthus wrote his essays. As evidence of this, the doubling time of the population is continuously growing shorter (5, 18). In 1650, the world population was approximately 500 million. The first billion was reached in 1830; the doubling time was about 180 years. Two billion was reached in 1930 for a doubling time of 100 years. Four billion was reached in 1976 for a doubling time of 46 years. In 1984, at the growth rate of 1.8%, it was found that only 39 more years would be needed to double the world population (5, 19). People born in 1960, the year world population reached 3 billion, will see the world population increase by about 5 billion in their lifetimes as 8 billion is expected to be reached between the years 2015 and 2020 (5, 20). These years represent a doubling time between 39 and 44 years.

Though this incredible growth may have surprised Malthus, whose writings appeared before the most significant effects of the industrial revolution, he probably would not have been surprised to learn of the many problems accompanying such growth. Problems such as "high illiteracy, ill health, malnutrition and heavy age dependency" plague many developing countries today (6, 269). Most of these countries "have been able to do little to raise standards of living and still experience widespread poverty" (6, 269). About 70 per cent of the world's population live in developing countries and due to high fertility and declining mortality in these regions, they "contribute about 4/5 of present world population growth" (6, 269).

When comparing statistics of the United States with those of Mexico, a developing country, the differences can be seen easily. The rate of growth in the United States from population statistics in 1975 was .86 while in Mexico it was an astonishing 3.25 (17, 224). The doubling time of the United States was 80 years as compared to the Mexican doubling time of only 21 years. 64.3 per cent of the population of the United States is between the ages of 15 and 64. In Mexico, this statistic was at only 50 per cent. These differences in the developed and the developing worlds exist today.

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